Government College of Engineering and Research
Avasari, Pune

Fundamental of Finite Element Analysis

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Unit 6
Dynamic Analysis

Unit Outcome

- General Dynamic Equation for free and forced vibration
- Formulation of undamped free vibration by eigenvalue method
- Determine natural frequency bar element by eigenvalue method

General Dynamic Equation for undamped free vibration

$$m\ddot{x} + kx = 0$$

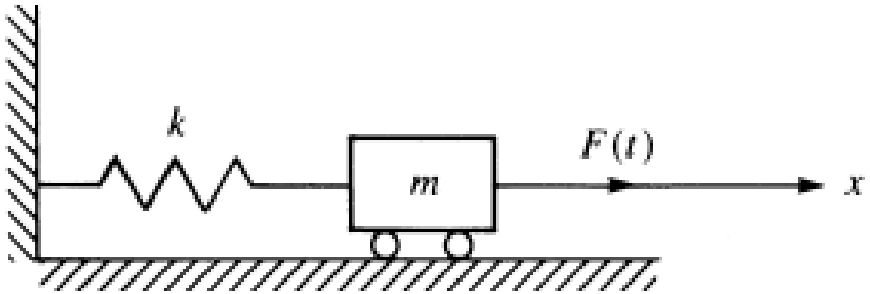
$$\text{or} \quad m\frac{d^2x}{dt^2} + kx = 0$$

General Dynamic Equation for forced vibration

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

Dynamics of a Spring-Mass System



Applying Newton's second law of motion, $f = ma$, to the mass, we obtain the equation of motion in the x direction as

$$F(t) - kx = m\ddot{x} \qquad (16.1.1)$$

where a dot over a variable denotes differentiation with respect to time; that is, $(\dot{\ }) = d(\ )/dt$. Rewriting Eq. (16.1.1) in standard form, we have

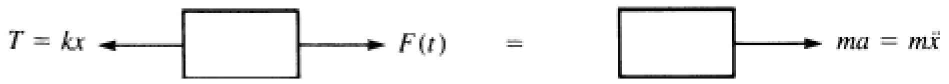
$$m\ddot{x} + kx = F(t) \qquad (16.1.2)$$

The homogeneous solution to Eq. (16.1.2) is the solution obtained when the right side is set equal to zero. A number of useful concepts regarding vibrations are obtained by considering this free vibration of the mass—that is, when $F(t) = 0$. Hence, defining

$$\omega^2 = \frac{k}{m} \tag{16.1.3}$$

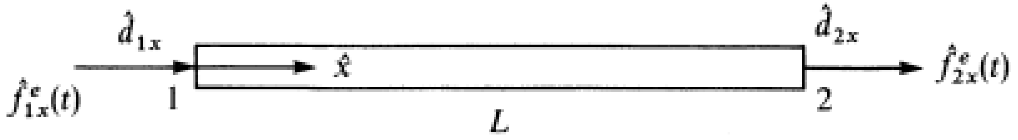
and setting the right side of Eq. (16.1.2) equal to zero, we have

$$\ddot{x} + \omega^2 x = 0 \tag{16.1.4}$$



where ω is called the **natural circular frequency** of the free vibration of the mass, expressed in units of radians per second or revolutions per minute (rpm). Hence, the natural circular frequency defines the number of cycles per unit time of the mass vibration. We observe from Eq. (16.1.3) that ω depends only on the spring stiffness k and the mass m of the body.

Direct Derivation of Bar Element



Again, we assume a linear displacement function along the $\hat{x}$ axis of the bar [see Eq. (3.1.1)]; that is, we let

$$\hat{u} = a_1 + a_2\hat{x} \tag{16.2.1}$$

As was shown in Chapter 3, Eq. (16.2.1) can be expressed in terms of the shape functions as

$$\hat{u} = N_1\hat{d}_{1x} + N_2\hat{d}_{2x} \tag{16.2.2}$$

where $$N_1 = 1 - \frac{\hat{x}}{L} \qquad N_2 = \frac{\hat{x}}{L} \tag{16.2.3}$$

Again, the strain/displacement relationship is given by

$$\{\varepsilon_x\} = \frac{\partial \hat{u}}{\partial \hat{x}} = [B]\{\hat{d}\}$$

where $$[B] = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \qquad \{\hat{d}\} = \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

and the stress/strain relationship is given by

$$\{\sigma_x\} = [D]\{\varepsilon_x\} = [D][B]\{\hat{d}\}$$

The bar is generally not in equilibrium under a time-dependent force; hence, $f_{1x} \neq f_{2x}$. Therefore, we again apply Newton's second law of motion, $\boldsymbol{f} = \boldsymbol{ma}$, to each node. In general, the law can be written for each node as "the external (applied) force f_x^e minus the internal force is equal to the nodal mass times acceleration." Equivalently, adding the internal force to the ***ma*** term, we have

$$\hat{f}^e_{1x} = \hat{f}_{1x} + m_1 \frac{\partial^2 \hat{d}_{1x}}{\partial t^2} \qquad \hat{f}^e_{2x} = \hat{f}_{2x} + m_2 \frac{\partial^2 \hat{d}_{2x}}{\partial t^2} \tag{16.2.7}$$

where the masses m_1 and m_2 are obtained by lumping the total mass of the bar equally at the two nodes such that

$$m_1 = \frac{\rho A L}{2} \qquad m_2 = \frac{\rho A L}{2} \tag{16.2.8}$$

$$\begin{Bmatrix} \hat{f}^e_{1x} \\ \hat{f}^e_{2x} \end{Bmatrix} = \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} + \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 \hat{d}_{1x}}{\partial t^2} \\ \dfrac{\partial^2 \hat{d}_{2x}}{\partial t^2} \end{Bmatrix}$$

Using Eqs. (3.1.13) and (3.1.14), we replace $\{\hat{f}\}$ with $[\hat{k}]\{\hat{d}\}$ in Eq. the element equations

$$\{\hat{f}^e(t)\} = [\hat{k}]\{\hat{d}\} + [\hat{m}]\{\ddot{\hat{d}}\}$$

where
$$[\hat{k}] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

is the bar element stiffness matrix, and

$$[\hat{m}] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is called the **lumped-mass matrix**. Also,

$$\{\ddot{\hat{d}}\} = \frac{\partial^2\{\hat{d}\}}{\partial t^2}$$

Consistent-mass Matrix

This mass matrix is called the *consistent-mass matrix* because it is derived from the same shape functions $[N]$ that are used to obtain the stiffness matrix $[\hat{k}]$. In general, $[\hat{m}]$ given by Eq. (16.2.19) will be a full but symmetric matrix. Equation (16.2.19) is

$$[\hat{m}] = \iiint_V \rho [N]^T [N] \, dV$$

$$[\hat{m}] = \iiint_V \rho \begin{Bmatrix} 1 - \dfrac{\hat{x}}{L} \\ \dfrac{\hat{x}}{L} \end{Bmatrix} \begin{bmatrix} 1 - \dfrac{\hat{x}}{L} & \dfrac{\hat{x}}{L} \end{bmatrix} dV$$

Simplifying Eq. (16.2.20), we obtain

$$[\hat{m}] = \rho A \int_0^L \begin{Bmatrix} 1 - \dfrac{\hat{x}}{L} \\ \dfrac{\hat{x}}{L} \end{Bmatrix} \begin{bmatrix} 1 - \dfrac{\hat{x}}{L} & \dfrac{\hat{x}}{L} \end{bmatrix} d\hat{x}$$

$$\{F(t)\} = [K]\{d\} + [M]\{\ddot{d}\}$$

$$[K] = \sum_{e=1}^{N} [k^{(e)}] \qquad [M] = \sum_{e=1}^{N} [m^{(e)}] \qquad \{F\} = \sum_{e=1}^{N} \{f^{(e)}\}$$

1.	**Bar Element**	1	$\frac{\rho A l}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$
2.	**Plane Truss Element**	2	$\frac{\rho A l}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\frac{\rho A l}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ & 2 & 0 & 1 \\ \text{Sym metric} & & 2 & 0 \\ & & & 2 \end{bmatrix}$
3.	**Three Noded CST Element**	2	$\frac{\rho A t}{3}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & \text{Symmetric} & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{bmatrix}$	$\frac{\rho A t}{12}\begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ & 2 & 0 & 1 & 0 & 1 \\ & & 2 & 0 & 1 & 0 \\ & \text{Symmetric} & & 2 & 0 & 1 \\ & & & & 2 & 0 \\ & & & & & 2 \end{bmatrix}$
4.	**Beam Element**	2	$\frac{\rho A l}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$

Formulation of Undamped free vibration by Eigenvalue Method

Governing Equation for Undamped Free Vibrations of Element :

$$m\ddot{u} + ku = 0$$

$$\therefore [m]\{\ddot{u}_N\} + [k]\{u_N\} = 0$$

Governing Equation for Undamped Free Vibration of Assembly :

$$[M]\{\ddot{U}_N\} + [K]\{U_N\} = 0$$

For simple harmonic motion,

$$\ddot{x} = -\omega^2 x$$

Hence, $\{\ddot{U}_N\} = -\omega^2\{U_N\}$

$$-\omega^2[M]\{U_N\} + [K]\{U_N\} = 0$$

$$[[K] - \omega^2[M]]\{U_N\} = 0$$

$$[[K] - \lambda[M]]\{U_N\} = 0$$

$$[[K] - \lambda[M]]\{U_N\} = 0$$

where, $\lambda = \omega^2$ = eigenvalue

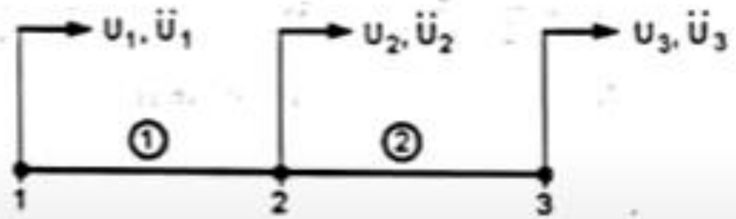
$[K]$ = global stiffness matrix

$[M]$ = global mass matrix

$\{U_N\}$ = global nodal displacement vector

Find the natural frequencies of longitudinal vibrations of the unconstrained stepped bar of cross-sectional areas A and 2A, having equal step lengths, as shown in

1. **Discretization of stepped bar :**



Element Connectivity

Element Number ⓔ	Global Node Number 'n' of	
	Local Node 1	Local Node 2
①	1	2
②	2	3

2. Element stiffness matrices :

Element 1 :

$$[k]_1 = \frac{A_1E_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2A \times E}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{AE}{L}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{array}\begin{array}{c} n \\ 1 \\ 2 \end{array} \text{ N/m}$$

Element 2 :

$$[k]_2 = \frac{A_2E_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{AE}{L}\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} n \\ 2 \\ 3 \end{array} \text{ N/m}$$

3. Global stiffness matrix :

$$[K] = [k]_1 + [k]_2$$

$$[K] = \frac{AE}{L}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 2 & -2 & 0 \\ -2 & (2+1) & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} n \\ 1 \\ 2 \\ 3 \end{array} = \frac{AE}{L}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} n \\ 1 \\ 2 \\ 3 \end{array} \text{ N/m}$$

Lumped mass matrix for bar element is,

$$[m] = \frac{\rho A l}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Consistent mass matrix for bar element is,

$$[m] = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

4. Consistent element mass matrices :

As consistent mass matrix approach is more accurate as compared to lumped mass matrix approach, consistent mass m is used.

- **Element 1 :**

$$[m]_1 = \frac{\rho A_1 l_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\rho \times 2A \times L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{A}{L}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} \frac{2\rho L^2}{3} & \frac{\rho L^2}{3} \\ \frac{\rho L^2}{3} & \frac{2\rho L^2}{3} \end{bmatrix} \end{array} \begin{array}{c} n \\ 1 \\ 2 \end{array} \text{kg}$$

- **Element 2 :**

$$[m]_2 = \frac{\rho A_2 l_2}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{A}{L}\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} \frac{\rho L^2}{3} & \frac{\rho L^2}{6} \\ \frac{\rho L^2}{6} & \frac{\rho L^2}{3} \end{bmatrix} \end{array} \begin{array}{c} n \\ 2 \\ 3 \end{array} \text{kg}$$

5. **Global consistent mass matrix :**

$$[M] = [m]_1 + [m]_2$$

$$[K] = \frac{A}{L}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{2\rho L^2}{3} & \frac{\rho L^2}{3} & 0 \\ \frac{\rho L^2}{3} & \left(\frac{2\rho L^2}{3}+\frac{\rho L^2}{3}\right) & \frac{\rho L^2}{6} \\ 0 & \frac{\rho L^2}{6} & \frac{\rho L^2}{3} \end{bmatrix} \end{array} \begin{array}{c} n \\ 1 \\ 2 \\ 3 \end{array} = \frac{A}{L}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{2\rho L^2}{3} & \frac{\rho L^2}{3} & 0 \\ \frac{\rho L^2}{3} & \rho L^2 & \frac{\rho L^2}{6} \\ 0 & \frac{\rho L^2}{6} & \frac{\rho L^2}{3} \end{bmatrix} \end{array} \begin{array}{c} n \\ 1 \\ 2 \\ 3 \end{array} \text{ kg}$$

6. **Global nodal displacement vector (Eigenvector) :**

$$\{U_N\} = \left\{ \begin{array}{c} U_1 \\ U_2 \\ U_3 \end{array} \right\} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \text{ m}$$

7. **Equation of eigenvalue and eigen vector :**

$$[[K] - \lambda[M]]\{U_N\} = 0$$

$$[[K] - \lambda[M]]\{U_N\} = 0$$

$$\left\{ \frac{AE}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{\lambda A}{L}\begin{bmatrix} \frac{2\rho L^2}{3} & \frac{\rho L^2}{3} & 0 \\ \frac{\rho L^2}{3} & \rho L^2 & \frac{\rho L^2}{6} \\ 0 & \frac{\rho L^2}{6} & \frac{\rho L^2}{3} \end{bmatrix} \right\} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = 0$$

$$\left\{ \frac{AE}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{AE}{L}\begin{bmatrix} \frac{2\lambda\rho L^2}{3E} & \frac{\lambda\rho L^2}{3E} & 0 \\ \frac{\lambda\rho L^2}{3E} & \frac{\lambda\rho L^2}{E} & \frac{\lambda\rho L^2}{6E} \\ 0 & \frac{\lambda\rho L^2}{6E} & \frac{\lambda\rho L^2}{3E} \end{bmatrix} \right\} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = 0$$

Take $\alpha = \dfrac{\lambda\rho L^2}{6E}$

$$\left\{\frac{AE}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{AE}{L}\begin{bmatrix} 4\alpha & 2\alpha & 0 \\ 2\alpha & 6\alpha & \alpha \\ 0 & \alpha & 2\alpha \end{bmatrix}\right\}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = 0$$

$$\frac{AE}{L}\begin{bmatrix} 2-4\alpha & -2-2\alpha & 0 \\ -2-2\alpha & 3-6\alpha & -1-\alpha \\ 0 & -1-\alpha & 1-2\alpha \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = 0$$

8. Specified boundary conditions :

It is an unconstrained bar. Hence, there are no specified boundary conditions.

Hence, equation (j) becomes,

$$\begin{bmatrix} 2(1-2\alpha) & -2(1+\alpha) & 0 \\ -2(1+\alpha) & 3(1-2\alpha) & -(1+\alpha) \\ 0 & -(1+\alpha) & (1-2\alpha) \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = 0$$

9. Determination of eigenvalue :

$$\begin{vmatrix} 2(1-2\alpha) & -2(1+\alpha) & 0 \\ -2(1+\alpha) & 3(1-2\alpha) & -(1+\alpha) \\ 0 & -(1+\alpha) & (1-2\alpha) \end{vmatrix} = 0$$

$$2(1-2\alpha)[3(1-2\alpha)^2-(1+\alpha)^2]+2(1+\alpha)[-2(1+\alpha)(1-2\alpha)-0] = 0$$

$$(1-2\alpha)[3-12\alpha+12\alpha-1-2\alpha-\alpha^2]-2(1+\alpha)[1-2\alpha+\alpha-2\alpha^2] = 0$$

$$(1-2\alpha)(11\alpha^2-14\alpha+2)+(2+2\alpha)(2\alpha^2+\alpha-1) = 0$$

$$(11\alpha^2-14\alpha+2-22\alpha^3+28\alpha^2-4\alpha)+(4\alpha^2+2\alpha-2+4\alpha^3+2\alpha^2-2\alpha) = 0$$

$$(-22\alpha^3 + 39\alpha^2 - 18\alpha + 2) + (4\alpha^3 + 6\alpha^2 - 2) = 0$$

$$(-18\alpha^3 + 45\alpha^2 - 18\alpha) = 0$$

$$9\alpha(-2\alpha^2 + 5\alpha - 2) = 0$$

$$\alpha(2\alpha^2 - 5\alpha + 2) = 0$$

$$\therefore \quad \alpha = 0 \quad \text{or} \quad (2\alpha^2 - 5\alpha + 2) = 0$$

$$\alpha = 0 \quad \text{or} \quad \alpha = \frac{5 \pm \sqrt{(-5)^2 - 4 \times 2 \times 2}}{2 \times 2}$$

$$\alpha = 0 \quad \text{or} \quad \alpha = \frac{5 \pm 3}{2}$$

$$\alpha = 0 \quad \text{or} \quad \alpha = 0.5 \quad \text{or} \quad \alpha = 2$$

$$\text{But } \alpha = \frac{\lambda \rho L^2}{6E}$$

$$\therefore \lambda = \frac{6\alpha E}{\rho L^2}$$

$$\lambda = 0 \quad \text{or} \quad \frac{6 \times 0.5 \times E}{\rho L^2} \quad \text{or} \quad \frac{6 \times 2 \times E}{\rho L^2}$$

$$\lambda = 0 \quad \text{o} \quad \frac{3E}{\rho L^2} \quad \text{or} \quad \frac{12E}{\rho L^2}$$

10. Determination of natural frequency :

$$\omega^2 = \lambda$$

$$\therefore \quad \omega^2 = 0 \quad \text{or} \quad \frac{3E}{\rho L^2} \quad \text{or} \quad \frac{12E}{\rho L^2}$$

$$\therefore \quad \omega = 0 \text{ rad/s} \quad \text{or} \quad \frac{1.73}{L}\sqrt{\frac{E}{\rho}} \text{ rad/s} \quad \text{or} \quad \frac{3.46}{L}\sqrt{\frac{E}{\rho}} \text{ rad/s}$$

$$\omega_1 = 0 \text{ rad/s} \; ; \quad \omega_2 = \frac{1.73}{L}\sqrt{\frac{E}{\rho}} \text{ rad/s and } \omega_3 = \frac{3.46}{L}\sqrt{\frac{E}{\rho}} \text{ rad/s}$$

$$f_1 = \frac{\omega_1}{2\pi} = 0 \text{ Hz}$$

and $$f_2 = \frac{\omega_2}{2\pi} = \frac{1.73}{2\pi \times L}\sqrt{\frac{E}{\rho}} = \frac{0.275}{L}\sqrt{\frac{E}{\rho}} = \text{Hz}$$

and $$f_3 = \frac{\omega_3}{2\pi} = \frac{3.46}{2\pi \times L}\sqrt{\frac{E}{\rho}} = \frac{0.55}{L}\sqrt{\frac{E}{\rho}} = \text{Hz}$$

$$f_1 = 0 \; ; \qquad f_2 = \frac{0.275}{L}\sqrt{\frac{E}{\rho}} \text{ Hz} \quad \text{and} \quad f_3 = \frac{0.55}{L}\sqrt{\frac{E}{\rho}} \text{ Hz}$$

1.	**Bar Element**	1	$\frac{\rho A l}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$	$\frac{\rho A l}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$
2.	**Plane Truss Element**	2	$\frac{\rho A l}{2}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$	$\frac{\rho A l}{6}\begin{bmatrix}2 & 0 & 1 & 0\\ & 2 & 0 & 1\\ \text{Sym metric} & & 2 & 0\\ & & & 2\end{bmatrix}$
3.	**Three Noded CST Element**	2	$\frac{\rho A t}{3}\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ & 1 & 0 & 0 & 0 & 0\\ & & 1 & 0 & 0 & 0\\ \text{Symmetric} & & & 1 & 0 & 0\\ & & & & 1 & 0\\ & & & & & 1\end{bmatrix}$	$\frac{\rho A t}{12}\begin{bmatrix}2 & 0 & 1 & 0 & 1 & 0\\ & 2 & 0 & 1 & 0 & 1\\ & & 2 & 0 & 1 & 0\\ \text{Symmetric} & & & 2 & 0 & 1\\ & & & & 2 & 0\\ & & & & & 2\end{bmatrix}$
4.	**Beam Element**	2	$\frac{\rho A l}{2}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$	$\frac{\rho A l}{420}\begin{bmatrix}156 & 22l & 54 & -13l\\ 22l & 4l^2 & 13l & -3l^2\\ 54 & 13l & 156 & -22l\\ -13l & -3l^2 & -22l & 4l^2\end{bmatrix}$

Estimate the natural frequencies of axial vibrations of bar shown in Fig. P.6.9.5(a), using both consistent as well as lumpe mass matrices and compare the results. The bar is having uniform cross-section with cross-sectional area $A = 50 \times 10^{-6}$ m length $L = 1.5$ m, modulus of elasticity $E = 2 \times 10^{11}$ N/m^2 and density $\rho = 7800$ kg/m^3. Model the bar by using two elements.

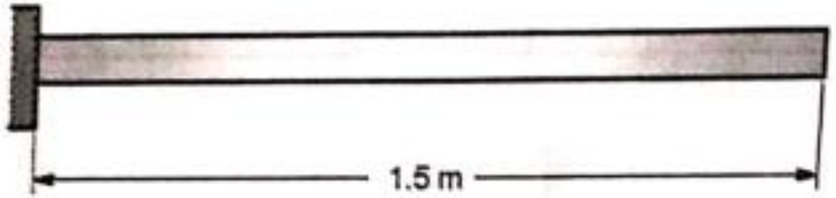


Fig. P. 6.9.5(a)

Solution :

Given : $A_1 = 50 \times 10^{-6}\ m^2$; $L = 1.5$ m ;

$E = 2 \times 10^{11}$ N/m^2 ; $\rho = 7800$ kg/m^3 ;

$l_1 = l_2 = l = \frac{L}{2} = \frac{1.5}{2} = 0.75$ m.

1 **Discretization of bar :**

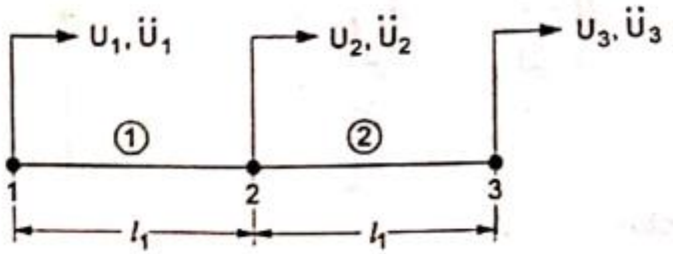


Fig. P. 6.9.5(b)

- The stepped bar is modelled with two bar elements, as shown in Fig. P. 6.9.5(b).
- **Element connectivity for stepped bar :**

Element Number ⓔ	Global Node Number 'n' of	
	Local Node 1	Local Node 2
①	1	2
②	2	3

Degrees of freedom of assembly (N) :

$$N = \text{D.O.F. per node} \times \text{Number of nodes in assembly} = 1 \times 3 = 3$$

Both the elements are identical. Hence, $[k]_1 = [k]_2$ and $[m]_1 = [m]_2$

- **Degrees of freedom of assembly (N) :**

$$N = \text{D.O.F. per node} \times \text{Number of nodes in assembly} = 1 \times 3 = 3$$

- Both the elements are identical. Hence, $[k]_1 = [k]_2$ and $[m]_1 = [m]_2$

Element stiffness matrices :

$$[k]_1 = [k]_2 = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ N/m}$$

Global stiffness matrix :

$$[K] = [k]_1 + [k]_2$$

$$[K] = \frac{AE}{l}\begin{bmatrix} 1 & -1 & 0 \\ -1 & (1+1) & -1 \\ 0 & -1 & 1 \end{bmatrix} = \frac{AE}{l}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} n \\ 1 \\ 2 \\ 3 \end{array} \text{ N/m}$$

Consistent element mass matrices :

$$[m]_1 = [m]_2 = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{ kg}$$

Global consistent mass matrix :

$$[M] = [m]_1 + [m]_2$$

$$[M] = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & (2+2) & 1 \\ 0 & 1 & 2 \end{bmatrix} = \frac{AE}{l}\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ & \begin{bmatrix} \frac{\rho l^2}{3E} & \frac{\rho l^2}{6E} & 0 \\ \frac{\rho l^2}{6E} & \frac{2\rho l^2}{3E} & \frac{\rho l^2}{6E} \\ 0 & \frac{\rho l^2}{6E} & \frac{\rho l^2}{3E} \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array} \text{ kg}$$

6. Global nodal displacement vector :

$$\{U_N\} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \text{ m}$$

7. Equation of eigenvalue and eigenvector :

$$[[K] - \lambda[M]]\{U_N\} = 0$$

$$\left\{\frac{AE}{l}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix} - \frac{\lambda AE}{l}\begin{bmatrix}\frac{\rho l^2}{3E} & \frac{\rho l^2}{6E} & 0\\ \frac{\rho l^2}{6E} & \frac{2\rho l^2}{3E} & \frac{\rho l^2}{6E}\\ 0 & \frac{\rho l^2}{6E} & \frac{\rho l^2}{3E}\end{bmatrix}\right\}\begin{Bmatrix}U_1\\ U_2\\ U_3\end{Bmatrix} = 0$$

$$\frac{AE}{l}\begin{bmatrix}\left(1-\frac{\lambda\rho l^2}{3E}\right) & -\left(1+\frac{\lambda\rho l^2}{6E}\right) & 0\\ -\left(1+\frac{\lambda\rho l^2}{6E}\right) & \left(2-\frac{2\lambda\rho l^2}{3E}\right) & -\left(1+\frac{\lambda\rho l^2}{6E}\right)\\ 0 & -\left(1+\frac{\lambda\rho l^2}{6E}\right) & \left(1-\frac{\lambda\rho l^2}{3E}\right)\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\end{Bmatrix} = 0$$

Take $\alpha = \frac{\lambda\rho l^2}{E}$

Substituting Equation (g) in Equation (f).

$$\frac{AE}{l}\begin{bmatrix}\left(1-\frac{\alpha}{3}\right) & -\left(1+\frac{\alpha}{6}\right) & 0\\ -\left(1+\frac{\alpha}{6}\right) & \left(2-\frac{2\alpha}{3}\right) & -\left(1+\frac{\alpha}{6}\right)\\ 0 & -\left(1+\frac{\alpha}{6}\right) & \left(1-\frac{\alpha}{3}\right)\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\end{Bmatrix} = 0$$

$$\begin{bmatrix} \left(2-\frac{2\alpha}{3}\right) & -\left(1+\frac{\alpha}{6}\right) \\ -\left(1+\frac{\alpha}{6}\right) & \left(1-\frac{\alpha}{3}\right) \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = 0$$

termination of eigenvalue :

From equation (i),

$$\begin{vmatrix} \left(2-\frac{2\alpha}{3}\right) & -\left(1+\frac{\alpha}{6}\right) \\ -\left(1+\frac{\alpha}{6}\right) & \left(1-\frac{\alpha}{3}\right) \end{vmatrix} = 0$$

$$\left(2-\frac{2\alpha}{3}\right)\left(1-\frac{\alpha}{3}\right)-\left(1+\frac{\alpha}{6}\right)^2 = 0$$

$$\left(2-\frac{2\alpha}{3}-\frac{2\alpha}{3}+\frac{2\alpha^2}{9}\right)-\left(1+\frac{\alpha}{3}+\frac{\alpha^2}{36}\right) = 0$$

$$\left(\frac{2\alpha^2}{9}-\frac{4\alpha}{3}+2-\frac{\alpha^2}{36}-\frac{\alpha}{3}-1\right) = 0$$

$$\left(\frac{7\alpha^2}{36}-\frac{5\alpha}{3}+1\right) = 0$$

$$7\alpha^2-60\alpha+36 = 0$$

$$\alpha = \frac{+60 \pm \sqrt{(-60)^2 - 4 \times 7 \times 36}}{2 \times 7}$$

$$\alpha = \frac{60 \pm 50.912}{14}$$

$$\therefore \alpha = 0.649 \quad \text{or} \quad 7.922$$

$$\text{But } \alpha = \frac{\lambda \rho l^2}{E} \text{ [From equation (g)]}$$

Substituting equations (g) in equation (j),

$$\frac{\lambda \rho l^2}{E} = 0.649 \text{ or } 7.922$$

$$\therefore \quad \frac{\lambda \times 7800 \times (0.75)^2}{2 \times 10^{11}} = 0.649 \text{ or } 7.922$$

$$\therefore \quad \lambda \times 2.19375 \times 10^{-8} = 0.649 \text{ or } 7.922$$

$$\therefore \quad \lambda = 29.59 \times 10^6 \text{ or } 361.11 \times 10^6$$

Determination of natural frequency :

$$\omega^2 = \lambda$$

$$\omega^2 = 29.59 \times 10^6 \quad \text{or} \quad 361.11 \times 10^6$$

$$\therefore \omega = 5439.67 \text{ rad/s} \quad \text{or} \quad 19002.9 \text{ rad/s}$$

$$\omega_1 = 5439.67 \text{ rad/s} \quad \text{and} \quad \omega_2 = 19002.9 \text{ rad/s}$$

(II) Lumped Matrix Method

1. **Discretization of bar :**

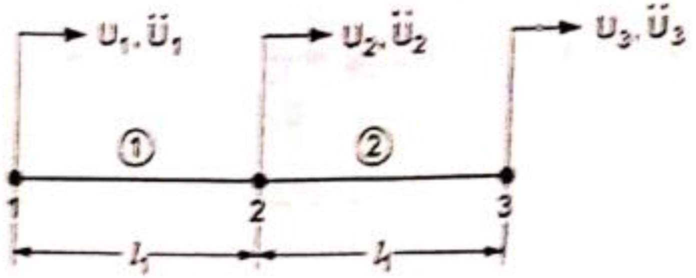


Fig. P. 6.9.5(b)

- The stepped bar is modelled with two bar elements, as shown in Fig. P. 6.9.5(b).

Element connectivity for stepped bar :

Table P. 6.9.5 : Element Connectivity

Element Number ⓔ	Global Node Number 'n' of	
	Local Node 1	Local Node 2
①	1	2
②	2	3

Degrees of freedom of assembly (N) :

$$N = \text{D.O.F. per node} \times \text{Number of nodes in assembly} = 1 \times 3 = 3$$

$$l_1 = l_2 = \frac{L}{2} = \frac{1.5}{2} = 0.75 \text{ m}$$

Both the elements are identical. Hence, $[k]_1 = [k]_2$ and $[m]_1 = [m]_2$

ment stiffness matrices :

$$[k]_1 = [k]_2 = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ N/m}$$

3. **Global stiffness matrix :**

$$[K] = [k]_1 + [k]_2$$

$$[K] = \frac{AE}{l}\begin{bmatrix} 1 & -1 & 0 \\ -1 & (1+1) & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \leftarrow n \\ \dfrac{AE}{l} & \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array} \text{N/m}$$

4. **Lumped element mass matrices :**

$$[m]_1 = [m]_2 = \frac{\rho Al}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{kg}$$

5. **Global lumped mass matrix :**

$$[M] = [m]_1 + [m]_2$$

$$[M] = \frac{\rho Al}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \leftarrow n \\ \dfrac{AE}{l} & \begin{bmatrix} \dfrac{\rho l^2}{2E} & 0 & 0 \\ 0 & \dfrac{\rho l^2}{E} & 0 \\ 0 & 0 & \dfrac{\rho l^2}{2E} \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array} \text{kg}$$

6. **Global nodal displacement vector :**

$$\{U_N\} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} m$$

7. **Equation of eigenvalue and eigenvector :**

$$[[K] - \lambda[M]]\{U_n\} = 0$$

$$\left\{\frac{AE}{l}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{\lambda AE}{l}\begin{bmatrix} \frac{\rho l^2}{2E} & 0 & 0 \\ 0 & \frac{\rho l^2}{E} & 0 \\ 0 & 0 & \frac{\rho l^2}{2E} \end{bmatrix}\right\}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = 0$$

$$\frac{AE}{l}\begin{bmatrix} \left(1-\frac{\lambda\rho l^2}{2E}\right) & -1 & 0 \\ -1 & \left(2-\frac{\lambda\rho l^2}{E}\right) & -1 \\ 0 & -1 & \left(1-\frac{\lambda\rho l^2}{2E}\right) \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = 0$$

$$\text{Take} \quad \alpha = \frac{\lambda \rho l^2}{E} \quad \text{... (q)}$$

Substituting Equation(q) in Equation (p),

$$\frac{AE}{l}\begin{bmatrix} \left(1-\frac{\alpha}{2}\right) & -1 & 0 \\ -1 & (2-\alpha) & -1 \\ 0 & -1 & \left(1-\frac{\alpha}{2}\right) \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = 0 \quad \text{... (r)}$$

8. **Specified boundary conditions :**

- At node 1, there is rigid support. Hence, $U_1 = 0$. As d.o.f. 1 is fixed, first row and first column can be eliminated from Equation (r)
- Hence, Equation (r) becomes,

$$\begin{bmatrix} (2-\alpha) & -1 \\ -1 & \left(1-\frac{\alpha}{2}\right) \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = 0$$

9. Determination of eigenvalue :

- From Equations (s),

$$\begin{vmatrix} (2-\alpha) & -1 \\ -1 & \left(1-\frac{\alpha}{2}\right) \end{vmatrix} = 0$$

$$(2-\alpha)\left(1-\frac{\alpha}{2}\right)-1 = 0$$

$$2-\alpha-\alpha+\frac{\alpha^2}{2}-1 = 0$$

$$\frac{\alpha^2}{2}-2\alpha+1 = 0$$

$$\alpha^2 - 4\alpha + 2 = 0$$

$$\alpha = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 2}}{2 \times 1}$$

$$\alpha = \frac{4 \pm 2.83}{2}$$

$$\therefore \quad \alpha = 0.583 \quad \text{or } 3.417$$

$$\text{But} \quad \alpha = \frac{\lambda \rho l^2}{E} \quad \text{[From equation (q)]}$$

Substituting equations (q) in equation (t),

$$\therefore \quad \frac{\lambda \rho l^2}{E} = 0.583 \text{ or } 3.417$$

$$\frac{\lambda \times 7800 \times (0.75)^2}{2 \times 10^{11}} = 0.583 \text{ or } 3.4147$$

$$\lambda \times 2.19375 \times 10^{-8} = 0.58 \text{ or } 3.414$$

$$\therefore \quad \lambda = 26.44 \times 10^6 \text{ or } 155.62 \times 10^6$$

10. Determination of natural frequency :

$$\omega^2 = \lambda$$

$$\omega^2 = 26.57 \times 10^6 \text{ or } 155.76 \times 10^6$$

$$\therefore \omega = 5141.6 \text{ rad/s or } 12480.4 \text{ rad/s}$$

$$\omega_1 = 5141.6 \text{ rad/s} \quad \text{and} \quad \omega_2 = 12480.4 \text{ rad/s}$$

(III) Comparison of Results

- **Consistent matrix method :**

$$\omega_1 = 5439.67 \text{ rad/s} \quad ; \quad \omega_2 = 19002.9 \text{ rad/s}$$

- **Lumped matrix method :**

$$\omega_1 = 5141.6 \text{ rad/s} \quad ; \quad \omega_2 = 12480.4 \text{ rad/s}$$

Thank You

For Your Attention